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BOUNDARY VALUE PROBLEM WITH A SPECTRAL PARAMETER

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Abstract. In this paper we consider one class of irregular boundary value problems for ordinary differential equations of third order. Their defective coercivity is investigated and the defect of coercivity is found and the area of defective coercivity is determined, which proves the existence of a unique solution of the problem. An estimation of the solution has obtained.

Keywords: Irregular boundary value problems, the defective coercivity, Fourier transform.
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1 Introduction

The theory of spectral problems for irregular differential operators, due to its theoretical and applied importance, is one of the intensively developing branches of modern mathematics and attracts the attention of many researchers interested in both the theory itself and its applications. The first thorough research in this direction was undertaken by the Ward (1925, 1927, 1932) and Stone (1927). They studied the expansion of smooth functions on root functions of the some irregular differential operatrs second and third orders.

The spectral problems related to higher order differential operators caused by irregular non-separable boundary conditions (i.e. including the coercivity on spectral parameter, completeness of the root functions system (specific and attachment functions system) and the problems of the separation functions according to such systems) is one of the current actual topics Birkhoff (1908); Birkhoff & Langer (1923); Aliev (2022); Gerhard (2015); Israfilov (2023, 2022); Iskenderova (2007); Seiferta (1951); Iskenderova (2007); Xu et al. (2023); Gasimov et al. (2010).

We consider the equation

$$\mathrm{L}_0(\lambda)\mathrm{u} = \lambda^3 u(x) + au'''(x) + bu''(x) = f(x), \quad x \in (0,1), \tag{1}$$

with boundary conditions

$$\mathrm{L}_{i0} u = \alpha_i u^{(m_i)}(0) + \beta_i u^{(m_i)}(1) = f_i, \tag{2}$$

where a, b, α_i, β_i, f_i - are complex numbers, $a \neq 0$, $|\alpha_i| + |\beta_i| \neq 0$, $i = 1,2,3$.

Introduce

$$\theta_0 = \begin{vmatrix} \alpha_1 w_1^{m_1} & \alpha_1 w_2^{m_1} & \beta_1 w_3^{m_1} \\ \alpha_2 w_1^{m_2} & \alpha_2 w_2^{m_2} & \beta_2 w_3^{m_2} \\ \alpha_3 w_1^{m_3} & \alpha_3 w_2^{m_3} & \beta_3 w_3^{m_3} \end{vmatrix},$$

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$$\theta_1 = \begin{vmatrix} \alpha_1 m_1 w_1^{m_1-1} & \alpha_1 w_2^{m_1} & \beta_1 w_3^{m_1} \\ \alpha_2 m_2 w_1^{m_2-1} & \alpha_2 w_2^{m_2} & \beta_2 w_3^{m_2} \\ \alpha_3 m_3 w_1^{m_3-1} & \alpha_3 w_2^{m_3} & \beta_3 w_3^{m_3} \end{vmatrix} + \begin{vmatrix} \alpha_1 w_1^{m_1} & \alpha_1 m_1 w_2^{m_1-1} & \beta_1 w_3^{m_1} \\ \alpha_2 w_1^{m_2} & \alpha_2 m_2 w_2^{m_2-1} & \beta_2 w_3^{m_2} \\ \alpha_3 w_1^{m_3} & \alpha_3 m_3 w_2^{m_3-1} & \beta_3 w_3^{m_3} \end{vmatrix} +$$

$$+ \begin{vmatrix} \alpha_1 w_1^{m_1} & \alpha_1 w_2^{m_1} & \beta_1 m_1 w_3^{m_1-1} \\ \alpha_2 w_1^{m_2} & \alpha_2 w_2^{m_2} & \beta_2 m_2 w_3^{m_2-1} \\ \alpha_3 w_1^{m_3} & \alpha_3 w_2^{m_3} & \beta_3 m_3 w_3^{m_3-1} \end{vmatrix},$$

where w_1=1, $w_2 = e^{\frac{2\pi i}{3}}$, $w_3 = e^{\frac{4\pi i}{3}}$ and denote the roots of the equation $a\omega^3 + 1 = 0$ as ω_j, $j = 1, 2, 3$. If

$$\underline{\omega} = min\left\{arg\omega_1,\ arg\omega_2,\ arg\omega_3 + \pi\right\},$$

$$\overline{\omega} = max\left\{arg\omega_1,\ arg\omega_2,\ arg\omega_3 + \pi\right\},$$

$\overline{\omega} - \underline{\omega} < \pi$, then $\underline{\omega} = \frac{\pi - arga}{3}$, $\overline{\omega} = \pi - \frac{arga}{3}$, where $-\pi <$ arg$a \leq \pi$.

The problem (1)-(2) will be irregular, if $\theta_0 = 0, \theta_1 \neq 0$ according to a definition in Naimark (1969)p.66.

Unlike regular boundary condition problems in Birkhof-Tamarkin sense, irregular spectral boundary problems are usually not coercive on the spectral parameter and the resolvent of the appropriate operator satisfy the condition $\|R(\lambda, \mathrm{A})\| \leq C|\lambda|^{-1+\alpha}$, $\alpha > 0$, when the spectral parameter is at a certain cone of the complex plane that does not include the imaginary axis.

Solvability of boundary value problems for a second order elliptic differential-operator equation with quadratic complex parameter in a separable Hilbert space was investigated in Aliev (2022). An application of the obtained abstract results to elliptic boundary problems is given there.

Birkhoff-irregular boundary value problems for quadratic ordinary differential pencils of the second order have been considered in Yakubov (1998b) also. The spectral parameter may appear in the boundary condition, the equation contains an abstract linear operator while the boundary conditions contain internal points of an interval and a linear functional. The corresponding spectral problem of two-fold completeness of root functions is established according to the isomorphism and coerciveness with a defect. As an application of the obtained results, an initial boundary value probruction of the coefficients of the differential equation by the Weyl matrix and by several spectra properties are studied in Bondarenko (2021). The uniqueness of solution for these inverse problems is proved there, by developing the method of spectral mappings and the results for the second-order differential operators with singular potentials and for the higher-order differential operators with regular coefficients.

The inverse spectral problem of recover the Sturm- Liouville operator with non-separated boundary conditions, one of which contains a spectral parameter are studied in Nabiev (2022). There a uniqueness of theorem is proved, a solving algorithm is constructed and sufficient conditions for solvability of inverse problem are obtained. Similar questions was investigated in Markus (1988) also. The inverse spectral problem of recovering the Sturm-Liouville operator with non-separated boundary conditionlem for second order parabolic equations is considered, and the completeness of the elementary solutions are proved.

The inverse spectral problems which consist of the reconsts, one of which contains a linear function of the spectral parameter was studied in Mammadova (2024). The spectrum of one boundary value problem, some sequence of signs, and some number are used as spectral data.

2 The isomorphism of the irregular operator

Theorem. If $a \neq 0$, $b \neq 0$, $\theta_0 = 0, \theta_1 \neq 0$, then $\forall\varepsilon > 0$ $\exists\delta > 0$, $\exists R_\varepsilon > 0$: $\forall\lambda$: $\frac{\pi}{6} + \frac{arga}{3} + \varepsilon < arg\lambda < \frac{\pi}{2} + \frac{arga}{3} - \varepsilon$, $|\lambda| > R_\varepsilon$, for the irregular operator

$$\mathcal{L}_0(\lambda)u{:}u \to (\mathrm{L}_0(\lambda)u,\ \mathrm{L}_{10}(\lambda)u,\ \mathrm{L}_{20}(\lambda)u,\ \mathrm{L}_{30}(\lambda)u)\,,$$

$\mathcal{L}_0(\lambda) : W_q^l(0,1) \to W_q^l(0,1) \dot{+} C^3$, with integer number $l \geq \max\{3, m_i+1\}$ and $q \in (1,\infty)$ is isomorphic and exists the estimation for the solution of the problem (1)-(2) and for these λ's:

$$\sum_{k=0}^{l} |\lambda|^{l-k-1} \|u\|_{W_q^k(0,1)} \leq C(\varepsilon) \left(\|f\|_{W_q^{l-3}(0,1)} + |\lambda|^{l-3} \|f\|_{L_q(0,1)} + \sum_{i=1}^{3} |\lambda|^{l-m_i-\frac{1}{q}} |f_i| \right), \quad (3)$$

where $C(\varepsilon)$ is constant.

Proof. The operator $\mathcal{L}_0(\lambda)$ is linear and continuous.

We look for the solution of the equation (1) in the form $u(x) = u_1(x) + u_2(x)$, where $u_1(x)$ is the solution of equation

$$a u_1'''(x) + b u_1''(x) + \lambda^3 u_1(x) = \check{f}(x), \quad x \epsilon R, \quad (4)$$

where $\check{f}(x) = \begin{cases} f(x), & x \in [0,1], \\ 0, & x \notin [0,1], \end{cases}$

and $u_2(x)$ is the solution of problem

$$a u_2'''(x) + b u_2''(x) + \lambda^3 u_2(x) = 0, \quad x \in (0,1), \quad (5)$$

$$L_{i_0} u_2 = -L_{i_0} u_1 + f_i, \quad i = 1,2,3.$$

We get from (5):

$$\left[a(i\sigma)^3 + b(i\sigma)^2 + \lambda^3 \right] F u_1 = F\widetilde{f}, \quad (6)$$

where F is Fourier transform. Obviously

$$a(i\sigma)^3 + b(i\sigma)^2 + \lambda^3 = a(i\sigma - \mu_1)(i\sigma - \mu_2)(i\sigma - \mu_3), \quad (7)$$

where $\mu_i,\ i = 1,2,3$ are roots of the corresponding characteristic equation

$$a\mu^3 + b\mu^2 + \lambda^3 = 0.$$

The general solution of equation (1) with $f(x) = 0$ for $|\lambda| \to \infty$ may be written as:

$$u(x) = c_1 e^{\mu_1 x} + c_2 e^{\mu_2 x} + c_3 e^{\mu_3 (x-1)}, \quad (8)$$

where c_i, $i = 1,2,3$ are arbitrary constants.

As for all λ: $\frac{\pi}{2} - \underline{\omega} + \varepsilon < arg\lambda < \frac{3\pi}{2} - \overline{\omega} - \varepsilon$, $|\lambda| \to \infty$ the estimates

$$|i\sigma - \mu_i| > C(\varepsilon)(|\sigma| + |\lambda|), i = 1,2,3, \quad (9)$$

hold, see [8] and then from (6) and (7) follows:

$$u_1^{(k)}(x) = F^{-1}(i\sigma)^k F u_1 = F^{-1}(i\sigma)^k \left[a(i\sigma)^3 + b(i\sigma)^2 + \lambda^3 \right]^{-1} F\tilde{f}. \quad (10)$$

From (7) and (9) follows, that

$$T_k(\sigma,\lambda) = \lambda^{3-k}(i\sigma)^k \left[a(i\sigma)^3 + b(i\sigma)^2 + \lambda^3 \right]^{-1}, k = 0,1,2,3,$$

when $\frac{\pi}{2} - \underline{\omega} + \varepsilon < arg\lambda < \frac{3\pi}{2} - \overline{\omega} - \varepsilon$, $|\lambda| \to \infty$,

are continuously differentiable functions of σ in R and the conditions hold

$$|T_k(\sigma,\lambda)| < C(\varepsilon), \ \left| \frac{\partial}{\partial\sigma} T_k(\sigma,\lambda) \right| \leq \frac{C(\varepsilon)}{|\sigma|}, k = \overline{0,3}.$$

According to [4, *p.*437], if $f\epsilon L_q(0,1)$, then $u_1(x)\epsilon W_q^3(R)$ from (10) for k = 0 is solution of (5) and

$$|\lambda|^{3-k}\left\|u_1^{(k)}\right\|_{L_q(0,1)} \leq \left\|F^{-1}T_k(\sigma,\lambda)F\tilde{f}\right\|_{L_q(R)} \leq C(\varepsilon)\left\|\tilde{f}\right\|_{L_q(R)} \leq C(\varepsilon)\|f\|_{L_q(0,1)}, k=\overline{0,3}.$$

Then, if $f(x)\epsilon W_q^{l-3}(0,1)$, then $u_1(x)\epsilon W_q^3(0,1)$ and the inequality

$$\sum_{k=0}^{l}|\lambda|^{l-k}\|u_1\|_{W_q^k(0,1)} \leq C\left(\varepsilon\right)\left(\|f\|_{W_q^{l-3}(0,1)}+|\lambda|^{l-3}\|f\|_{L_q(0,1)}\right) \tag{11}$$

holds, which may be proved by induction.

Now we can show the existence of a unique solution of problem (5) for arbitrary complex f_i, $i=1,2,3$, belonging to $W_q^l(0,1)$ and evaluate this solution.
The coefficients of the general solution of the equation (5) hace the form

$$C_j = D^{-1}(\lambda)\sum_{\mathrm{i}=1}^{3}a_{\mathrm{ji}}(\lambda)(-L_{i0}\mathrm{u}_1+f_i),$$

where $a_{\mathrm{ji}}(\lambda)=[\theta_{ji}+R_{ji}(\lambda)]\,\lambda^{\sum_{\mathrm{k}\neq\mathrm{i}}\mathrm{m_k}}$, θ_{ji} complex numbers $j,i=1,2,3$, $R_{ji}(\lambda)\to 0$ when

$$\frac{\pi}{6}+\frac{arga}{3}+\varepsilon<arg\lambda<\frac{\pi}{2}+\frac{arga}{3}-\varepsilon\ ,|\lambda|\to\infty.$$

Then problem (5) has a unique solution:

$$u_2(x)=\sum_{j=1}^{2}D^{-1}(\lambda)\sum_{\mathrm{i}=1}^{3}\left[\theta_{ji}+R_{ji}(\lambda)\right](-L_{i0}\mathrm{u}_1+f_i)\lambda^{\sum_{\mathrm{k}\neq\mathrm{i}}\mathrm{m_k}}\ \mathrm{e}^{\mu_{\mathrm{j}}x}+$$

$$+D^{-1}(\lambda)\sum_{\mathrm{i}=1}^{3}\left[\theta_{3i}+R_{3i}(\lambda)\right](-L_{i0}u_1+f_i)\lambda^{\sum_{\mathrm{k}\neq\mathrm{i}}\mathrm{m_k}}\ \mathrm{e}^{\mu_3(x-1)}, \tag{12}$$

$$\frac{\pi}{6}+\frac{arga}{3}+\varepsilon<arg\lambda<\frac{\pi}{2}+\frac{arga}{3}-\varepsilon\ ,|\lambda|\to\infty,$$

where $D(\lambda)$ is the determinant of the system for determining the coefficients c_i, $i=1,2,3$ in the general solution (8) having the form:
$\mathrm{D}(\lambda)=\left[\theta_0+\frac{b}{3a}\theta_1\lambda^{-1}+O(\lambda^{-2})\right](\delta_0\lambda)^{\sum_{\mathrm{i}=1}^{3}\mathrm{m_i}}+\mathrm{R}(\lambda)$, where $R(\lambda)\to 0$ when $\frac{\pi}{6}+\frac{arga}{3}+\varepsilon<arg\lambda<\frac{\pi}{2}+\frac{arga}{3}-\varepsilon$, $|\lambda|\to\infty$, satisfying

$$\left\|u_2^{(k)}\right\|_{L_q(0,1)} \leq C(\varepsilon)\sum_{i=1}^{3}\left(|L_{i0}\mathrm{u}_1|+|f_i|\right)|\lambda|^{-m_i+k+1-\frac{1}{q}}, k\geq 0, \tag{13}$$

when $\frac{\pi}{6}+\frac{arga}{3}+\varepsilon<arg\lambda<\frac{\pi}{2}+\frac{arga}{3}-\varepsilon$, $|\lambda|\to\infty$.
For estimation $|L_{i0}\mathrm{u}_1|$, $i=1,2,3$, we may use the inequality from [3, p.145]:

$$\left\|u^{(j)}\right\|_{C[0,1]} \leq C\left(h^{1-\chi}\left\|u^{(l)}\right\|_{L_q(0,1)}+h^{-\chi}\|u\|_{L_q(0,1)}\right),$$

where $0\leq \mathrm{j}<l$, $0<\mathrm{h}<\mathrm{h}_0$, $\chi=\left(j+\frac{1}{q}\right):l$. For $\mathrm{h}=|\lambda|^{-l}$ according go (11) we have

$$|L_{i0}\mathrm{u}_1| \leq C\|u_1\|_{C^{m_i}[0,1]} \leq C(\varepsilon)|\lambda|^{-l+m_i+\frac{1}{q}}\left(\|f\|_{W_q^{l-3}(0,1)}+|\lambda|^{l-3}\|f\|_{L_q(0,1)}\right),$$

$$\frac{\pi}{6}+\frac{arga}{3}+\varepsilon<arg\lambda<\frac{\pi}{2}+\frac{arga}{3}-\varepsilon\ ,|\lambda|\to\infty.$$

By substituting these estimations in (13) we have

$$|\lambda|^{l-k-1}\left\|u_2^{(k)}\right\|_{L_q(0,1)} \leq C(\varepsilon)\left(\|f\|_{W_q^{l-3}(0,1)}+|\lambda|^{l-3}\|f\|_{L_q(0,1)}+\sum_{i=1}^{3}|\lambda|^{l-m_i-\frac{1}{q}}|f_i|\right).$$

From this inequality and from (11) we get estimation (3) for the the solution of problem (1)-(2).

3 Conclusion

The discussed the effectiveness of different strategies in the study of spectral properties of irregular boundary problems. In the paper this problem is solved by using a methodology different from the traditional methods in the study of spectral properties of regular boundary problems. So, unlike the works where the method of constructing Green's function for regular boundary value problems is applied, here the method of Fourier transform is used and then theorems on the Fourier multipliers is applied. In differ from the regular case, the evolution of the solution does not decrease with respect to the parameter λ. The defective coercivity of the considered issue is proved and the area of defective coercivity is determined. The obtained results can be used for the investigation of the similar problems for the different boundary problems.

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